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ОБЛАСТІ АМПЛІТУДНОГО ТА ФАЗОВОГО РЕЗОНАНСУ В ЕЛЕКТРИЧНИХ МОСТОВИХ ЛАНЦЮГАХ

Анотація. *Робота присвячена дослідженню амплітудно-частотних характеристик двополюсника у вигляді мостового ланцюга з двома активними елементами - конденсатором і котушкою індуктивності в протилежних плечах. У таку схему укладаються два поширені ланцюги: реальний (з втратами на пасивних елементах) паралельний коливальний контур і реальний послідовний коливальний контур. Незважаючи на їх широке застосування в радіотехніці та електротехніці, точна частотна поведінка таких схем до останнього часу залишалася не дослідженою. На практиці досі застосовуються різні емпіричні та наближені методи, у яких питання їх застосування та точності (оцінка похибки) залишаються відкритими. Крім того, чисельні методи не дозволяють провести якісний аналіз процесів, що протікають у таких ланцюгах без точних формул для частотних характеристик ланцюга. Дана робота продовжує розпочате нами дослідження мостових двополюсників з активними елементами. Знайдені точні умови існування амплітудного та фазового резонансу, які раніше були невідомі та узагальнюють вже відомі окремі випадки.*

Ключові слова: *мостова схема, амплітудно-частотна характеристика, фазово-частотна характеристика, реальний паралельний RLC-контур, реальний послідовний RLC-контур, амплітудний резонанс, фазовий резонанс.*

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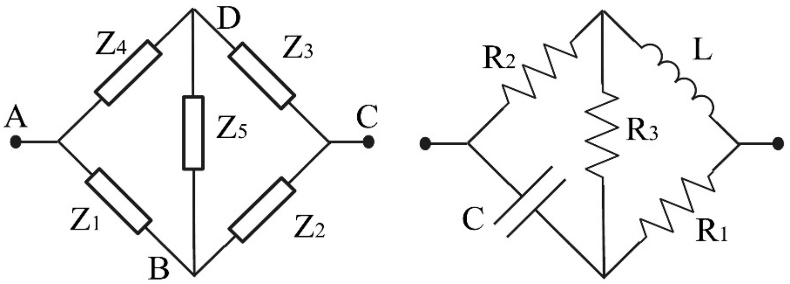
FREQUENCY DOMAINS OF AMPLITUDE AND PHASE RESONANCE IN ELECTRICAL BRIDGE NETWORKS

Abstract. *The work is devoted to the study of the amplitude-frequency characteristics of a two-terminal network in the form of a bridge circuit with two active elements—a capacitor and an inductance coil in opposite arms. Within the framework of such a circuit, two widely used circuits are combined: a real (with losses on passive elements) parallel oscillatory circuit and a real series oscillatory circuit. Despite their widespread use in radio engineering and electrical engineering, the exact frequency behaviour of such circuits has remained unexplored until recently. In practice, various empirical and approximate methods are still used, in which the issues of their application and accuracy (error estimation) remain open. In addition, numerical methods do not allow for a qualitative analysis of the processes occurring in such circuits without accurate formulas for the frequency characteristics of the circuit. This work continues our research on bridge two-terminal networks with active elements. We have found the exact conditions for the existence of amplitude and phase resonance, which were previously unknown and generalize already known individual cases.*

Keywords: *bridge circuit, amplitude-frequency characteristic, phase-frequency characteristic, real parallel RLC-circuit, real series RLC-circuit, amplitude resonance, phase resonance.*

1. **Introduction.** Bridge circuits are fundamental components in electrical engineering, widely used for measuring impedance, analysing resonance phenomena, and designing filters [1, 7, 8]. Understanding the conditions under which amplitude and phase resonance occur in such circuits is essential for optimizing their performance. In this study, a generalized bridge circuit is analysed, and the exact conditions defining the domains of resonance existence are determined. The analysis is based on admittance and impedance transformations that reveal the circuit's internal symmetries and self-duality properties.



Source: compiled by the author based on [1].

Fig. 1. **A two-terminal network in the form of a bridge and its special case with two active elements**

Let us consider a bridge circuit in the form of a two-terminal network shown in Fig. 1. An alternating voltage with frequency ω is applied to the input A and output C of the circuit. The current flowing through the circuit depends on the impedances of the elements Z_1, Z_2, Z_3, Z_4, and Z_5. The current is determined by the input impedance Z_{AC} (or admittance Y_{AC}). We will use the formulas for the admittance and impedance of the bridge, which can be obtained using the delta-star or star-delta transformation. A detailed analysis of such a circuit was first initiated in [5], where formulas for the impedance and admittance of the circuit were written in a form that reflects the existing symmetries.

$$Z_{AC} = \frac{Z_1Z_2(Z_3+Z_4)+Z_3Z_4(Z_1+Z_2)+Z_5(Z_1+Z_2)(Z_3+Z_4)}{(Z_1+Z_4)(Z_2+Z_3)+Z_5(Z_1+Z_2+Z_3+Z_4)} \tag{1}$$

$$Y_{AC} = \frac{Y_1Y_2(Y_3+Y_4)+Y_3Y_4(Y_1+Y_2)+Y_5(Y_1+Y_4)(Y_2+Y_3)}{(Y_1+Y_2)(Y_3+Y_4)+Y_5(Y_1+Y_2+Y_3+Y_4)} \tag{2}$$

Consider the bridge shown in Fig. 1(right) with resistors R_1, R_2, R_3, and inductance $L\neq 0$ and capacitance $C\neq 0$ in the opposite arms. We assume that not all R_i are equal to zero. Then the impedances of the arms and the diagonal of the square are as follows: $Z_1 = \frac{1}{j\omega C}$, $Z_2 = R_1$, $Z_3 = j\omega L$, $Z_4 = R_2$, $Z_5 = R_3$. We will investigate the impedance of this circuit, in the formula of which we will make such substitutions of variables that allow us to move from five independent parameters defining the circuit to three dimensionless parameters: $x = \omega\sqrt{CL}$, $a = R_1\sqrt{C/L}$, $b = R_2\sqrt{C/L}$, $c = R_3\sqrt{C/L}$. The new variable $x = \omega/\omega_0$, where $\omega_0 = 1/\sqrt{CL}$ is the resonance frequency of an ideal parallel or series LC circuit, is often called the reduced frequency, and the parameters *a*, *b*, *c* are quality factors (Q-factors). Such variables were introduced to analyse resonance phenomena in RLC circuits and various types of duality in [3, 4, 6]. In the new parameters, the input impedance takes the form:

$$Z_{AB} = \sqrt{\frac{L}{C}} \cdot f(x,a,b,c) \tag{3}$$

where $$f(x,a,b,c) = \frac{(a+b+c+abc)+j\left(a(b+c)x - b(a+c)/x\right)}{(1+ab+ac+bc)+j\left((b+c)x-(a+c)/x\right)} \tag{4}$$

The constant factor $\sqrt{L/C}$ is frequency-independent, and therefore, to describe the frequency characteristics of our circuit, it is sufficient to study only the function $f(x,a,b,c)$.

We introduce a new variable $w = \omega\sqrt{CL}\sqrt{\frac{b+c}{a+c}} = x\sqrt{\frac{b+c}{a+c}}$, which we will call the reduced bridge frequency. In terms of this variable, all our formulas acquire the simplest form. We also introduce two parameters $N^2 = \frac{(1+ab+ac+bc)^2}{(a+c)(b+c)}$ and $M^2 = \frac{(a+b+c+abc)^2}{(a+c)(b+c)}$, which can be considered as analogs of the concept of quality factor for bridge circuits. In the new notation, we obtain the function that will be studied further:

$$f(w,a,b,c) = \frac{M + j(aw - b/w)}{N + j(w - 1/w)} \quad (5)$$

2. Phase Resonance Analysis. Phase resonance occurs when the input voltage and current are in phase, resulting in zero phase shift. This happens when the imaginary part of the impedance vanishes: $\mathrm{Im}(f(w,a,b,c)) = 0$. Based on the derived expressions (5), the phase resonance frequency *w* is given by the formula

$$w_0(a,b,c)^2 = \frac{(a+c)(1-b^2)}{(b+c)(1-a^2)},\ x_0(a,b,c)^2 = \frac{(a+c)^2(1-b^2)}{(b+c)^2(1-a^2)},\ a \neq 1,\ b \neq 1 \quad (6)$$

Substituting this value into the impedance expression $f(w,a,b,c)$ yields the amplitude at resonance frequency

$$f(w_0,a,b,c) = f(x_0,a,b,c) = \frac{M}{N} = \frac{a+b+c+abc}{1+ab+ac+bc} \quad (7)$$

The necessary and sufficient condition for phase resonance existence requires the right-hand side of (6) to be positive, leading to specific relationships $(1-b^2)(1-a^2) > 0$ between parameters a and b: $0 < a < 1$ and $0 < b < 1$ or $a > 1$ and $b > 1$. The regions of phase resonance existence on the parameter plane a, b are illustrated in Fig. 3. The phase resonance frequency $x_0(a,b,c)$ and the amplitude at that frequency $f(x_0,a,b,c)$ exhibit two symmetry relations as defined in equations (8) and (9).

$$x_0(1/a,1/b,1/c) = x_0(a,b,c),\ x_0(b,a,c) = 1/x_0(a,b,c) \quad (8)$$

$$f(x_0,b,a,c) = f(x_0,a,b,c),\ f(x_0,1/a,1/b,1/c) = \frac{1}{f(x_0,a,b,c))} \quad (9)$$

Notably, the presence or absence of phase resonance does not depend on the diagonal BD resistance R_3 of the bridge.

3. Amplitude Resonance. The amplitude resonance frequency is defined as the frequency at which the amplitude of the current passing through the circuit reaches a maximum or minimum (current or voltage resonance). To find it, we calculate the squared modulus of the function $f(w,a,b,c)$:

$$F(w,a,b,c) = \left| f(w,a,b,c) \right|^2 = \frac{w^2M^2 + \left(aw^2 - b\right)^2}{w^2N^2 + \left(w^2 - 1\right)^2} \quad (10)$$

To find the extremum point of this function, we look for the root of its derivative. After differentiation, we obtain the equation for the extremum point:

$$F(w,a,b,c) = \frac{w^2M^2 + \left(aw^2 - b\right)^2}{w^2N^2 + \left(w^2 - 1)\right)^2} = \frac{a^2w^4 - b^2}{w^4 - 1} \tag{11}$$

After simple transformations, we arrive at the following biquadratic equation:

$$w^4(a^2N^2 - M^2 + 2a(b-a)) - 2w^2(b^2 - a^2) - (b^2N^2 - M^2 + 2b(a-b)) = 0 \tag{12}$$

Equation (12) possesses dualities inherited from the original bridge. These dualities arise from the substitutions $a \leftrightarrow b$ and inversion of all parameters $a \leftrightarrow 1/a$, $b \leftrightarrow 1/b$, $c \leftrightarrow 1/c$. If (12) is written as

$$k_4(a,b,c)w^4 + k_2\left(a,b,c\right)w^2 + k_0(a,b,c) = 0 \tag{13}$$

then $M(a,b,c) = M(b,a,c)$, $N(a,b,c) = N(b,a,c)$ (14)

$$k_2(a,b,c) = -k_2(b,a,c),\ k_4(a,b,c) = -k_0(b,a,c) \tag{15}$$

This means that the roots satisfy the condition

$$w_0^2(a,b,c) = \frac{1}{w_0^2(b,a,c)} \tag{16}$$

The condition

$$M^2(1/a,1/b,1/c) = \frac{1}{ab}N^2(b,a,c).\ N^2(1/a,1/b,1/c) = \frac{1}{ab}M^2(b,a,c) \tag{17}$$

means that when the parameters are inverted, equation (12) is multiplied by a constant, and its roots remain unchanged.

$$w_0^2(1/a,1/b,1/c) = w_0^2(b,a,c) \tag{18}$$

The discriminant D of equation (12) can be represented as

$$D = k_2^2(a,b,c) - 4k_4(a,b,c)k_0(a,b,c) = \frac{4(1-ab)^2}{(a+c)(b+c)} \cdot \left(c^2(1+ab+2a^2)(1+ab+2b^2) + c(3(1+ab)+2(a^2+b^2)) + (ab^2+2a+b)(a^2b+2b+a)\right) \tag{19}$$

From this representation, it follows that the discriminant is always positive (for $ab \neq 1$), which means that the quadratic equation always has roots. However, we are only interested in the positive root since it $w_0^2(a,b,c)$ corresponds to a physical frequency. From the origin of this equation as the condition for the extrema of $\left|f(x,a,b,c)\right|$ and from its dualities, it follows that this quadratic equation can have either two negative roots or one positive and one negative. A positive root exists, and hence an amplitude resonance point exists, only if $k_4(a,b,c)k_0(a,b,c) < 0$ That means

$$(a^2N^2 - M^2 + 2a(b-a))(b^2N^2 - M^2 + 2b(a-b)) > 0 \tag{20}$$

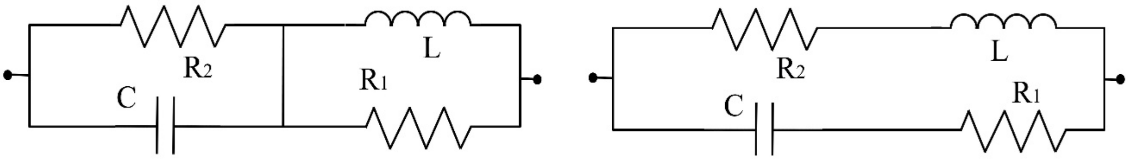
Returning to the parameters and simplifying by positive factors, condition (20) can be written as $\varphi(a,b,c)\cdot\psi(b,a,c)>0$

$$\varphi(a,b,c)=(a^4-1)(b+c)+2a(ab-1)(1+ac) \tag{21}$$

$$\psi(a,b,c)=\varphi(b,a,c)=(b^4-1)(a+c)+2b(ab-1)(1+bc) \tag{22}$$

Thus, the domain of existence of amplitude resonance in the parameter space (*a*, *b*, *c*) is determined by the conditions

$$\begin{cases}\varphi(a,b,c)>0\\ \varphi(b,a,c)>0\end{cases} \text{or} \begin{cases}\varphi(a,b,c<0\\ \varphi(b,a,c)<0\end{cases} \tag{23}$$



Source: compiled by the author based on [1].

Fig. 2. **Series (left, *c* = 0) and parallel (right, *c* =∞) circuits as special cases of the bridge circuit**

It should be noted that all the derived formulas are stable with respect to limiting transitions. They remain valid when the parameters (a,b,c) approach zero or infinity. In particular, when $c\to\infty$ our bridge circuit transforms into a real (lossy) series RLC circuit, and when $c\to 0$ – into a real parallel RLC circuit (see Fig. 2). For these circuits, equation (12) allows an elegant symmetric solution, first found in [3, 6] for the parallel circuit. In [2], the conditions for the existence of amplitude resonance for such a circuit were first derived, and the corresponding existence domains in the parameter space (*a*, *b*) were plotted. In [4, 5], it was shown how, through duality, the results for the parallel circuit can be extended to the series circuit, yielding the corresponding formulas. In the derived formulas, dual expressions appearing in the expansion of the discriminant D (19) of the quadratic equation play a key role:

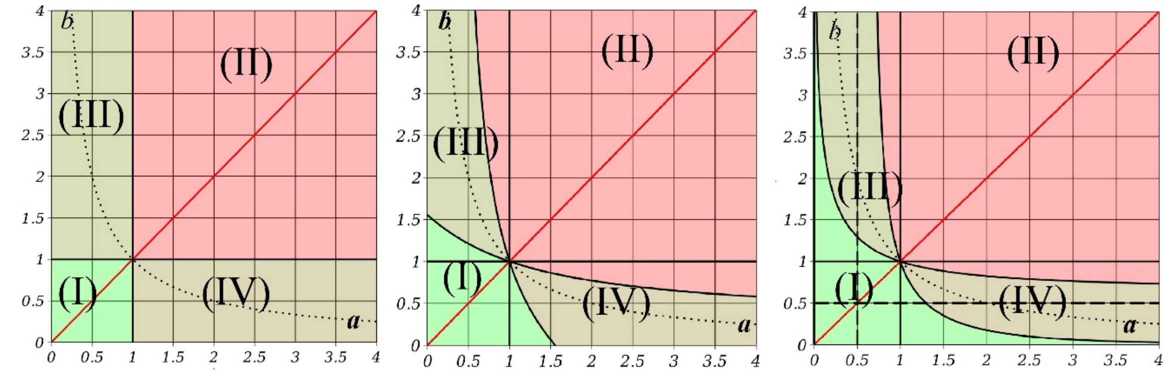
$$\left(1+ab+2a^2\right),\ \left(1+ab+2b^2\right),\ \left(ab^2+2a+b\right),\ \left(a^2b+2b+a\right) \tag{23}$$

$$x_0(a,b,\infty)^2=\frac{\sqrt{1+ab+2b^2}-b^2\sqrt{1+ab+2a^2}}{\sqrt{1+ab+2a^2}-a^2\sqrt{1+ab+2b^2}} \tag{24}$$

$$\begin{aligned}x_0(a,b,0)^2&=x_0(1/a,1/b,\infty)^2=\frac{\sqrt{1+1/(ab)+2/b^2}-1/b^2\sqrt{1+1/(ab)+2/a^2}}{\sqrt{1+1/(ab)+2/a^2}-1/a^2\sqrt{1+1/(ab)+2/b^2}}=\\ &=\frac{a\sqrt{a}}{b\sqrt{b}}\frac{\left(b\sqrt{ab}\sqrt{ab^2+2a+b}-\sqrt{a^2b+2b+a}\right)}{\left(a\sqrt{ab}\sqrt{a^2b+2b+a}-\sqrt{ab^2+2a+b}\right)}\end{aligned} \tag{25}$$

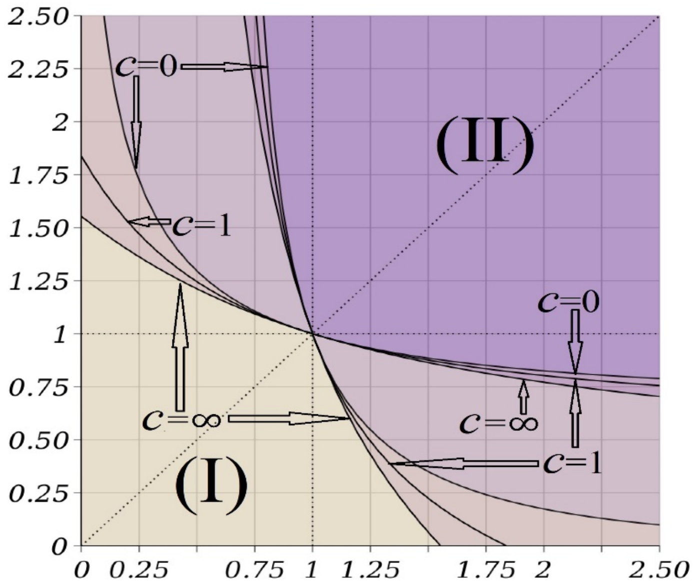
In Fig. 3 and Fig. 4, domains on the parameter planes (a,b) are shown where amplitude resonance exists or does not exist. On the left side of Fig. 3, the domains (I), (II) show the presence of phase resonance, while domains (III), (IV) show its absence. In the middle of Fig. 3, the domains of existence and nonexistence of amplitude resonance of the parallel RLC circuit (the special case of the bridge for $c = \infty$) are shown. The boundaries of these domains are defined by equations

$$\begin{cases} \varphi(a,b,\infty) = (a^4 - 1) + 2a^2(ab - 1) = 0 \\ \psi(a,b,\infty) = \varphi(b,a,\infty) = (b^4 - 1) + 2b^2(ab - 1) = 0 \end{cases} \quad (26)$$



Source: author's development using a computer program.

Fig. 3. **Domains of existence (I) and (II) and non-existence (III) and (IV) of phase (left) and amplitude resonance. In the center, $c = \infty$; on the right, $c = 0$**



Source: author's development using a computer program.

Fig. 4. **Regions of existence (I), (II) of amplitude resonance for arbitrary c with boundary curves for c = 0 and c = ∞**

The intersection points of the boundaries with the coordinate axes are at $a = \sqrt{1+\sqrt{2}}$ and $b = \sqrt{1+\sqrt{2}}$. On the right side of Fig. 3, the regions of existence and nonexistence of amplitude resonance of the series RLC circuit (the special case of the bridge for c = 0) are shown. The boundaries of these regions are determined by equations

$$\begin{cases} \varphi(a,b,0) = (a^4 - 1)b + 2a(ab - 1) = 0 \\ \psi(a,b,0) = \varphi(b,a,0) = (b^4 - 1)a + 2b(ab - 1) = 0 \end{cases} \quad (27)$$

These curves have asymptotes at $a = \sqrt{\sqrt{2}-1}$, $b = \sqrt{\sqrt{2}-1}$ and asymptotically approach the coordinate axes. In Fig. 4, the domains (I), (II) where amplitude resonance exists for arbitrary *c* are shown. As a typical example, the value *c* = 1 is taken. The boundaries of the resonance existence domains are determined by equations (21), (22) and are located between the curves for *c* = 0 and *c* = ∞. As seen in Fig. 4, when the parameter *c* decreases from ∞ to 0, the resonance existence region (I) expands. It should also be noted that in domains (I) the function $F(w,a,b,c) = \left| f(w,a,b,c) \right|^2$ has a maximum, corresponding to current resonance, while in domains (II) the modulus of the impedance has a minimum, corresponding to voltage resonance. In domains (III) the modulus of the impedance increases monotonically, and in domains (IV) it decreases monotonically.

4. Special and limiting cases. A particular point in the study of the frequency behaviour of the bridge circuit is the point $a = 1; b = 1$. At this point, the impedance becomes a constant function, independent of frequency. In the neighbourhood of this point, the resonance properties of the circuit become unpredictable – this is the bifurcation point of the resonance frequency. Approaching this point along different paths can lead to any resonance frequency value. The above formulas in this case become indeterminate.

If $a = b \neq 1$, all the formulas become absolutely symmetric with respect to substitution $a \leftrightarrow b$ and yield a unit reduced frequency of phase and amplitude resonance $x_0 = 1; \omega_0 = 1/\sqrt{CL}$.

For a balanced bridge $Z_1 Z_2 = Z_2 Z_4$, in our parameters we obtain the condition $a \cdot b = 1$. One can verify that in this case neither phase nor amplitude resonance occurs for any value of *c*. The curve $a \cdot b = 1$ always lies in the middle of domains (III) and (IV) for any value of *c*. In Fig. 3 it is shown as a dashed line.

5. Conclusion. The presented analysis establishes a unified basis for studying the frequency-dependent behavior of generalized bridge circuits. As a result of our research, we have obtained new, previously unknown results concerning the frequency behavior of bridge circuits. The derived conditions for amplitude and phase resonance determine the exact boundaries between resonant and non-resonant regions. The introduced dimensionless parameters allow us to obtain a compact analytical representation and emphasize the self-duality and symmetry inherent in the system. We show how the properties of self-duality and self-symmetry of the circuit are reflected in the obtained exact formulas. These results naturally extend to classical RLC circuits as limiting cases of the bridge configuration.

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